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CONTRIBUTION OF FIRST-ORDER MODULI DIFFERENCES TO DILATANT TRANSFORMATION TOUGHENING

X. HUANG and B. L. KARIHALOO
School of Civil and Mining Engineering, The University of Sydney, NSW, Australia 2006

and

W. B. FRASER
School of Mathematics and Statistics, The University of Sydney, NSW, Australia 2006

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Abstract—A perturbation expansion is used to investigate the influence of small differences in the elastic moduli between tetragonal and monoclinic zirconia precipitates upon the toughening induced by the dilatational component of this phase transformation. As expected this influence is insignificant to within the first order differences in the moduli, but the perturbation technique reveals that the fracture toughness is not simply a superposition of separate contributions from the dilatation and the moduli mismatch. It also reveals that the joint effect of the dilatation and moduli mismatch is qualitatively different from that predicted on the basis of the concept of effective dilatational strain.

1. INTRODUCTION

The phenomenon of transformation toughening in ceramic materials such as zirconia has received considerable attention in recent years and a number of theoretical analyses of the toughening mechanism have appeared (McMeeking and Evans, 1982; Budiansky *et al.*, 1983; Rice, 1985; Rose, 1987; Amazigo and Budiansky, 1988). In all of these theoretical models it is assumed that the elastic constants of the transformed particles are identical to those of the untransformed matrix material and in the case of toughened zirconia this is very nearly true. However even in this material there is a small difference between the elastic constants of the tetragonal and monoclinic phases. In this paper a perturbation expansion, that exploits the smallness of this difference in elastic constants, will be used to make an approximate estimation of the effect of this difference on the fracture toughness of the material.

As expected the influence of lowest order moduli differences is negligible, but the perturbation technique reveals two rather unexpected features of the solution. First, it shows that, even to the lowest order, the fracture toughness cannot be assumed to be a simple superposition of contributions from dilatation and moduli mismatch considered in isolation from each other. Secondly, it shows that the joint effect of the two is qualitatively different from the prediction based on the concept of effective dilatational strain (McMeeking, 1986). In view of the well-known similarity between the crack tip shielding by transformation and micro-crack induced dilatation, the above features are likely to carry over to the micro-cracking problem (Hutchinson, 1987).

The particular problem studied here is the plane strain model for steady-state crack growth investigated by McMeeking and Evans (1982) and Budiansky *et al.* (1983). It will be assumed that the elasticity tensors† $C_{\alpha\beta\gamma\delta}$ of the untransformed material (t-ZrO_2) and $\bar{C}_{\alpha\beta\gamma\delta}$ of the composite material (t-ZrO_2 + m-ZrO_2) consisting of particles that have undergone a mean stress-induced dilatant transformation embedded in a matrix of untransformed material in the neighbourhood of the crack are isotropic. In common with most works, the shear strains induced by the phase transformation will not be included in the analysis, although it is likely that they can have a significant influence on the solution (Lambropoulos,

† Greek subscripts take on values 1, 2 with the usual summation convention for repeated subscripts.

1986; Karihaloo and Huang, 1989). In order to make any progress it will be further assumed that $C_{\alpha\beta\gamma\delta}$ and $\bar{C}_{\alpha\beta\gamma\delta}$ are proportional so that

$$\bar{C}_{\alpha\beta\gamma\delta} = (1+\epsilon)C_{\alpha\beta\gamma\delta}. \tag{1}$$

As will be seen below, this assumption requires that Poisson's ratios ν, $\bar{\nu}$ of the untransformed precipitates and the composite material be equal. Thus

$$C_{\alpha\beta\gamma\delta} = 2\mu\left\{\frac{\nu}{1-2\nu}\delta_{\alpha\beta}\delta_{\gamma\delta}+\tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\beta\gamma}\delta_{\alpha\delta})\right\}, \tag{2}$$

$$\bar{C}_{\alpha\beta\gamma\delta} = 2\bar{\mu}\left\{\frac{\bar{\nu}}{1-2\bar{\nu}}\delta_{\alpha\beta}\delta_{\gamma\delta}+\tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\beta\gamma}\delta_{\alpha\delta})\right\}, \tag{3}$$

with

$$\bar{\mu} = (1+\epsilon)\mu, \quad \text{and} \quad \bar{\nu} = \nu.$$

The effective shear modulus $\bar{\mu}$ for the composite material in the neighbourhood of the crack can be calculated from the moduli μ and μ^{t} of the untransformed and transformed materials, respectively, by use of Hill's self-consistent method [eqns (17) and (18) in Hill (1965)]. In the case of toughened zirconia with volume fractions of the matrix and transforming phase 0.7 and 0.3, shear moduli μ = 78.93 GPa and μ^{t} = 96.29 GPa, bulk moduli κ = 143.06 GPa and κ^{t} = 174.54 GPa and Poisson's ratio $\nu \approx \nu^{t} = 0.267$ (Green *et al.*, 1989) the moduli of the composite are $\bar{\mu}$ = 84.08 GPa and $\bar{\kappa}$ = 151.73 GPa. This leads to the value $\epsilon = 0.065$ for the small parameter.

2. THE MATHEMATICAL FORMULATION

Let the x, y plane D contain a semi-infinite crack coincident with the negative x-axis $y = 0$, $x \leqslant 0$ subject to a remote mode I loading that would induce a stress intensity factor (S.I.F.) K^{L} at the crack tip in the absence of transformation. Let Ω be the region of steady-state transformed material surrounding the crack consisting of a parallel sided wake region of width $2H$ behind a small region of transformation ahead of the crack tip bounded by a smooth curve C. The rest of the D plane exterior to Ω will be designated by $D-\Omega$ (Fig. 1).

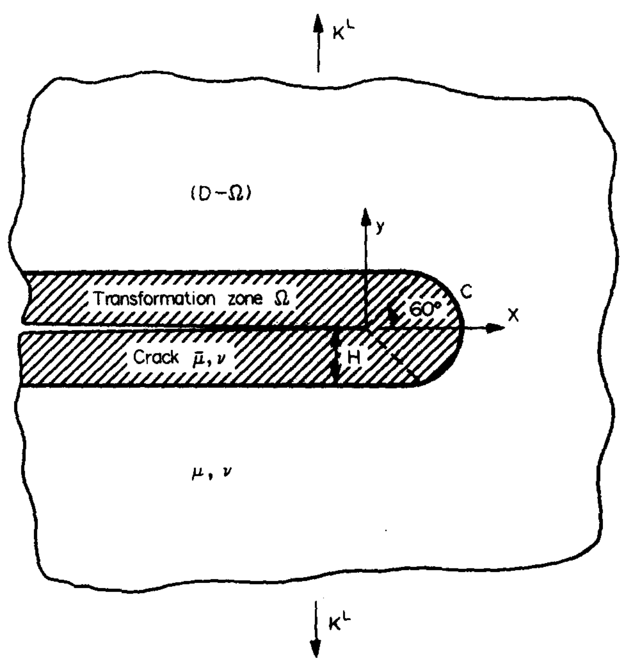

Fig. 1. A region Ω of steady-state transformed material whose shear modulus is slightly different from the rest of the region $D-\Omega$ as a result of dilatant phase transformation.

It will be assumed that the concentration c of transformed particles is constant throughout Ω. The plane strain $\varepsilon^T_{\alpha\beta}$ due to transformation is related to the stress-free dilatation θ^T_p that would occur in an unconstrained particle by the equation (Budiansky *et al.*, 1983)

$$\varepsilon^T_{\alpha\beta} = \tfrac{1}{3}(1+\nu)c\theta^T_p\delta_{\alpha\beta}, \tag{4}$$

where $c\theta^T_p$ is the volumetric transformation strain. The concept of effective transformation strain introduced by McMeeking (1986) for transforming composites in which the elastic properties of the transforming particles differ from those of the matrix would require that c in eqn (4) be replaced by an effective coefficient $\bar{c}$. For purely dilatant transformation strains, $\bar{c} = \{\kappa^t(\bar{\kappa}-\kappa)\}/\{\bar{\kappa}(\kappa^t-\kappa)\}$. For the zirconia composition under consideration, the effective dilatational strain would be $\bar{c}\theta^T_p$, with $\bar{c} = 0.3168$.

If $\varepsilon_{\alpha\beta}$ is the total strain then the stress is given by

$$\sigma_{\alpha\beta} = \begin{cases} C_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} & \text{in } D-\Omega, \\ \bar{C}_{\alpha\beta\gamma\delta}(\varepsilon_{\gamma\delta}-\varepsilon^T_{\gamma\delta}) & \text{in } \Omega. \end{cases} \tag{5}$$

The equilibrium equations are

$$\begin{aligned} (C_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta})_{,\beta} &= 0 \quad \text{in } D-\Omega, \\ [\bar{C}_{\alpha\beta\gamma\delta}(\varepsilon_{\gamma\delta}-\varepsilon^T_{\gamma\delta})]_{,\beta} &= 0 \quad \text{in } \Omega, \end{aligned} \tag{6}$$

and continuity of surface tractions across the boundary $\partial\Omega$ of the transformed region gives

$$C_{\alpha\beta\gamma\delta}\varepsilon^{\mathrm{OUT}}_{\gamma\delta}n_\beta = \bar{C}_{\alpha\beta\gamma\delta}(\varepsilon^{\mathrm{IN}}_{\gamma\delta}-\varepsilon^T_{\gamma\delta})n_\beta, \tag{7}$$

where n_β is the outward normal to the boundary of the transformed region, assumed positive if pointing from the inside (IN) to the outside (OUT) of this region.

The perturbation scheme is straightforward; all the dependent variables are expanded in power series in the small parameter:

$$\sigma_{\alpha\beta} = \sigma^0_{\alpha\beta} + \epsilon\sigma^1_{\alpha\beta} + \cdots,$$

and so on. When these expansions are substituted into the above equations and the coefficients of like powers of ϵ on both sides of the resulting equations are equated the following hierarchy of perturbation equations is obtained:

The $O(1)$ equations are

$$C_{\alpha\beta\gamma\delta}\varepsilon^0_{\gamma\delta,\beta} = \begin{cases} 0 & \text{in } D-\Omega, \\ C_{\alpha\beta\gamma\delta}\varepsilon^T_{\gamma\delta,\beta} & \text{in } \Omega, \end{cases}$$

with

$$C_{\alpha\beta\gamma\delta}[\varepsilon^0_{\gamma\delta}]^{\mathrm{OUT}}_{\mathrm{IN}}n_\beta = -C_{\alpha\beta\gamma\delta}\varepsilon^T_{\gamma\delta}n_\beta \tag{8}$$

on the boundary $\partial\Omega$.

The $O(\epsilon)$ equations are

$$C_{\alpha\beta\gamma\delta}\varepsilon^1_{\gamma\delta,\beta} = \begin{cases} 0 & \text{in } D-\Omega, \\ C_{\alpha\beta\gamma\delta}(\varepsilon^T_{\gamma\delta,\beta}-\varepsilon^0_{\gamma\delta,\beta}) & \text{in } \Omega, \end{cases}$$

with

$$C_{\alpha\beta\gamma\delta}[\varepsilon^1_{\gamma\delta}]^{\mathrm{OUT}}_{\mathrm{IN}}n_\beta = -C_{\alpha\beta\gamma\delta}(\varepsilon^T_{\gamma\delta}-\varepsilon^{0\,\mathrm{IN}}_{\gamma\delta})n_\beta \tag{9}$$

on the boundary $\partial\Omega$.

From eqns (8) it can be seen that the $O(1)$ strain $\varepsilon^0_{\alpha\beta}$ is due to a distribution of body force $F^0_\alpha = -C_{\alpha\beta\gamma\delta}\varepsilon^{\mathrm{T}}_{\gamma\delta,\beta}$ in Ω, and surface traction $T^0_\alpha = C_{\alpha\beta\gamma\delta}\varepsilon^{\mathrm{T}}_{\gamma\delta}n_\beta$ on the surface $\partial\Omega$ plus the strain due to the external load. Rice (1985) [see also Karihaloo and Huang (1989)] has shown how the weight function h_α can be used to calculate the change in S.I.F. due to the transformation:

$$K^0 = \iint_\Omega F^0_\alpha h_\alpha \,\mathrm{d}A + \int_{\partial\Omega} T^0_\alpha h_\alpha \,\mathrm{d}s.$$

When the expressions for F^0_α, and T^0_α are substituted into this expression and the divergence theorem is used to reduce the second integral above the final result is

$$K^0 = \iint_\Omega C_{\alpha\beta\gamma\delta}\varepsilon^{\mathrm{T}}_{\gamma\delta}h_{\alpha,\beta}\,\mathrm{d}A. \tag{10}$$

The $O(\epsilon)$ change in the S.I.F. can be calculated by the same method from eqns (9). The result is

$$\begin{aligned} K^1 &= \iint_\Omega C_{\alpha\beta\gamma\delta}(\varepsilon^{\mathrm{T}}_{\gamma\delta} - \varepsilon^0_{\gamma\delta})h_{\alpha,\beta}\,\mathrm{d}A \\ &= K^0 - \iint_\Omega C_{\alpha\beta\gamma\delta}\varepsilon^0_{\gamma\delta}h_{\alpha,\beta}\,\mathrm{d}A, \end{aligned} \tag{11}$$

and the total S.I.F. at the crack tip is

$$K_{\mathrm{TIP}} = K^{\mathrm{L}} + K^0 + \epsilon \mathrm{K}^1 + \mathrm{O}(\epsilon^2). \tag{12}$$

In order to calculate the above integrals the following complex quantities will now be introduced: complex independent variables:

$$z = x + \mathrm{i}y, \quad \bar{z} = x - \mathrm{i}y;$$

the complex plane strain weight function:

$$h = h_1 + \mathrm{i}h_2 = \frac{1}{2\sqrt{2\pi}(1-\nu)}\left\{\frac{-\chi}{2\sqrt{z}} + \frac{1}{\sqrt{\bar{z}}} - \frac{z+\bar{z}}{4\bar{z}\sqrt{\bar{z}}}\right\}, \tag{13}$$

where $\chi = 3 - 4\nu$, and the complex displacement:

$$w^0 = u^0_1 + \mathrm{i}u^0_2.$$

When expressions (2) and (4) are inserted into eqn (10) to give

$$K^0 = \frac{2\mu(1+\nu)c\theta^{\mathrm{T}}}{3(1-2\nu)}\iint_\Omega h_{\alpha,\alpha}\,\mathrm{d}A,$$

followed by use of the result $h_{\alpha,\alpha} = 2\,\mathrm{Re}\,\{\partial h/\partial z\}$, the final result is

$$K^0 = \frac{\mu(1+\nu)c\theta^{\mathrm{T}}}{6\sqrt{2\pi}(1-\nu)} \iint_{\Omega} \left\{ \frac{1}{z^{3/2}} + \frac{1}{\bar{z}^{3/2}} \right\} \mathrm{d}A.$$

Integration with respect to x further reduces this result to a contour integral around the curved boundary C of the transformed region ahead of the crack tip:

$$K^0 = -\frac{\mu(1+\nu)c\theta^{\mathrm{T}}}{3\sqrt{2\pi}(1-\nu)} \int_C \left\{ \frac{1}{\sqrt{z}} + \frac{1}{\sqrt{\bar{z}}} \right\} \mathrm{d}y. \tag{14}$$

If the effect of the transformation on the location of the boundary C is neglected, i.e. only the far-field stresses due to S.I.F. K^{L} are taken into account, then the shape of C is given by (McMeeking and Evans, 1982; Budiansky *et al.*, 1983)

$$\frac{(1+\nu)K^{\mathrm{L}}}{3\sqrt{2\pi}\sigma_{\mathrm{m}}^{\mathrm{c}}} \left\{ \frac{1}{\sqrt{z}} + \frac{1}{\sqrt{\bar{z}}} \right\} = 1, \quad -\frac{\pi}{3} \leqslant \arg(z) \leqslant \frac{\pi}{3}, \tag{15}$$

where $\sigma_{\mathrm{m}}^{\mathrm{c}}$ is the critical mean stress that induces the tetragonal to monoclinic phase transformation.

On this boundary the integral on the right of eqn (14) can be calculated exactly to give

$$K^0 = -\frac{\omega K^{\mathrm{L}}}{4\sqrt{3}\pi}, \tag{16}$$

where

$$\omega = \frac{2\mu(1+\nu)^2 c\theta_{\mathrm{p}}^{\mathrm{T}}}{(1-\nu)\sigma_{\mathrm{m}}^{\mathrm{c}}}, \tag{17}$$

is the parameter introduced by Amazigo and Budiansky (1988) which is a measure of the density of transformation in region Ω. Scrutiny of Fig. 2 in their paper shows that this result is a good approximation up to ω values of about 5.

In the method proposed by McMeeking (1986) for binary transforming composites, the expression for K^0 retains the form of eqn (16), but in the definition of ω the matrix elastic constants and c must be replaced with the composite elastic constants and $\bar{c}$, respectively. The corresponding density of transformation will be designated $\bar{\omega}$.

When expression (2) is inserted into the integral in eqn (11) it becomes

$$I = \iint_{\Omega} C_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}^{0} h_{\alpha,\beta}\, \mathrm{d}A = 2\mu \iint_{\Omega} \left\{ \frac{\nu}{1-2\nu} \varepsilon_{\alpha\alpha}^{0} h_{\beta,\beta} + \varepsilon_{\alpha\beta}^{0} h_{\alpha,\beta} \right\} \mathrm{d}A. \tag{18}$$

In order to calculate this integral it is necessary to find $\varepsilon_{\alpha\beta}^{0}$ which can be calculated from the complex displacement w^0. Either the weight function method of Rice (1985) or the method of Rose (1987) can be used to do this. In either case the result is

$$w^0(z,\bar{z}) = \frac{K^{\mathrm{L}}}{2\mu\sqrt{2\pi}} \left\{ \chi\sqrt{z} - \frac{z+\bar{z}}{2\sqrt{\bar{z}}} \right\} + \frac{(1+\nu)c\theta^{\mathrm{T}}}{24\pi(1-\nu)} \iint_{\Omega} \left\{ \frac{4}{\bar{z}-\bar{z}_0} + \phi(z,\bar{z},z_0) + \phi(z,\bar{z},\bar{z}_0) \right\} \mathrm{d}A_0, \tag{19}$$

where

$$\phi(z,\bar{z},z_0) = \frac{2}{\sqrt{z_0}(\sqrt{z_0}+\sqrt{\bar{z}})} - \frac{\chi}{\sqrt{z_0}(\sqrt{z_0}+\sqrt{z})} - \frac{1}{(\sqrt{z_0}+\sqrt{\bar{z}})^2} - \frac{z+\bar{z}}{2\sqrt{\bar{z}z_0}(\sqrt{z_0}+\sqrt{\bar{z}})^2}.$$

In terms of $w^0(z,\bar{z})$ and $h(z,\bar{z})$ the integral (18) is

$$I = 2\mu \int\int_\Omega \left\{ \frac{2}{(1-2\nu)} \operatorname{Re}\left\{\frac{\partial h}{\partial z}\right\} \operatorname{Re}\left\{\frac{\partial w^0}{\partial z}\right\} + 2\operatorname{Re}\left\{\frac{\partial \bar{h}}{\partial z}\frac{\partial w^0}{\partial \bar{z}}\right\}\right\} \mathrm{d}A. \tag{20}$$

3. RESULTS AND DISCUSSION

Some of the integrals in eqn (20) can be calculated analytically, while the rest have to be evaluated numerically. In analytical integration care has to be exercised to isolate any non-integrable singularity at the crack tip. This is done by surrounding the latter with a circular core with the matrix moduli as suggested by Hutchinson (1987). It is of course now essential to realize that the corresponding value of the integral is *not* a contribution to the desired K_{TIP}, but to the singular fields within the inner circular core. To obtain the correct contributions to K_{TIP}, the procedure proposed by Hutchinson (1987) for the corresponding micro-crack shielding problem is adopted here. Of course, this procedure is only approximate for our purposes as it ignores the interaction effects, but to the lowest order differences in moduli the error is expected to be negligible.

Referring to the Appendix, it follows that

$$2\mu \int\int_\Omega \frac{2}{(1-2\nu)} \operatorname{Re}\left\{\frac{\partial h}{\partial x}\right\} \operatorname{Re}\left\{\frac{\partial w^0}{\partial z}\right\} \mathrm{d}A = -\frac{(1-2\nu)}{16\pi(1-\nu)} K^{\mathrm{L}}(3\sqrt{3}-2\pi)$$
$$-\frac{\omega(1-2\nu)}{288\pi^2(1-\nu)} K^{\mathrm{L}}\left[-1.1188+6\sqrt{3}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)-(5\sqrt{3}\pi+\tfrac{4}{3}\pi^2)-1.6915\right], \tag{21}$$

$$2\mu \int\int_\Omega 2\operatorname{Re}\left\{\frac{\partial \bar{h}}{\partial z}\frac{\partial w^0}{\partial \bar{z}}\right\} \mathrm{d}A = -\frac{K^{\mathrm{L}}}{8\pi(1-\nu)}\left[\left(\frac{\sqrt{3}}{2}\right)^3 - \frac{\pi}{6}\right]$$
$$+\frac{\omega}{192\pi^2(1-\nu)} K^{\mathrm{L}}\left[2\sqrt{3}\left\{\left(\frac{\sqrt{3}}{2}\right)^3-\frac{\pi}{6}\right\}-8.7426\right]. \tag{22}$$

Finally, from eqns (11), (20)–(22) the $O(\epsilon)$ change in S.I.F. (for $\nu = 0.267$) can be evaluated:

$$K^1 = K^0 - 0.0070K^{\mathrm{L}} - 0.0022\omega K^{\mathrm{L}}, \tag{23}$$

where K^0 is given by eqn (16).

The total S.I.F. at the crack tip is ($\epsilon = 0.065$)

$$K_{\mathrm{TIP}} = K^{\mathrm{L}} + \epsilon(-0.0481\omega K^{\mathrm{L}} - 0.0070K^{\mathrm{L}}) - 0.0459\omega K^{\mathrm{L}}$$
$$= K^{\mathrm{L}} - 0.0031\omega K^{\mathrm{L}} - 0.0005K^{\mathrm{L}} - 0.0459\omega K^{\mathrm{L}}.$$

As previously noted, the above K_{TIP} is not strictly the desired value. It has to be corrected (albeit approximately) by the procedure adopted by Hutchinson (1987). As a result, the first three terms get divided by $(1+\frac{5}{8}\delta_1-\frac{3}{4}\delta_2)$, where $\delta_1 = ((\mu/\bar{\mu})-1)/(1-\nu)$ and $\delta_2 = \nu\delta_1$. [The last term is the (correct) contribution from dilatation alone.] For the zirconia composition under study, $\delta_1 = -0.0835$, $\delta_2 = -0.0223$, so that to the lowest order differences in moduli

$$K_{\mathrm{TIP}} = 1.0362K^{\mathrm{L}} - 0.0032\omega K^{\mathrm{L}} - 0.0459\omega K^{\mathrm{L}}. \tag{24}$$

The first term in the right-hand side of eqn (24) is the contribution of moduli changes alone and this contribution would appear to be deleterious to the overall toughening of the material under study. The second term is the contribution from the joint effect of the phase transformation and the lowest order moduli changes induced by this transformation. It is seen that this effect is synergistic. The last term is the contribution from dilatation alone. It is instructive to compare eqn (24) with the corresponding expression for micro-crack induced shielding [Hutchinson (1987), eqn (3.9)]

$$\frac{K_{\mathrm{TIP}}}{K^{\mathrm{L}}} = 1 + (k_1 - \tfrac{5}{8})\delta_1 + (k_2 + \tfrac{3}{4})\delta_2 - 0.0459\omega,$$

where ω is still defined by (17) but with $c\theta_{\mathrm{p}}^{\mathrm{T}}$ reinterpreted as the dilatation due to the formation of micro-cracks, and

$$k_1 = \frac{1}{32\pi}\int_0^{\pi} (11\cos\theta + 8\cos 2\theta - 3\cos 3\theta)\ln[R(\theta)]\,\mathrm{d}\theta,$$

$$k_2 = -\frac{1}{2\pi}\int_0^{\pi} (\cos\theta + \cos 2\theta)\ln[R(\theta)]\,\mathrm{d}\theta.$$

$R(\theta)$ refers to the boundary C. It is easily verified that k_1 and k_2 correspond to the leading terms in (21) and (22), i.e. the terms independent of ω. To within constant multipliers involving μ and ν the integrals contained in these leading terms are (A3) and (A7) of the Appendix.

For the steadily growing crack (Fig. 1), $k_1 = -0.0166$, $k_2 = -0.0433$ [Hutchinson (1987), eqn (4.9)], so that

$$\frac{K_{\mathrm{TIP}}}{K^{\mathrm{L}}} = 1.0350 - 0.0459\omega. \tag{25}$$

Comparison of (24) and (25) shows that, even to the lowest order differences in moduli, the fracture toughness is not a simple superposition of individual contributions from the dilatation and the moduli mismatch, but that there is a coupling between the two.

Equation (24) also shows that, contrary to the prediction based on the concept of effective dilatational strain (McMeeking, 1986), the net shielding effect in a binary transforming composite cannot be calculated by a simple replacement of ω with $\bar{\omega}$ in the definition of K^0 [eqn (16)]. To see this, let us calculate $\bar{\omega}$ from ω [eqn (17)] after replacing μ, ν and c with $\bar{\mu}$, $\bar{\nu}$ and $\bar{c}$, respectively. For the zirconia composition under study this gives $\bar{\omega} = 1.1246\omega$, so that the toughness ratio according to the effective dilatational strain approach is

$$\frac{K_{\mathrm{TIP}}}{K^{\mathrm{L}}} = 1 - 0.0459\bar{\omega} = 1 - 0.0516\omega, \tag{26}$$

whereas the perturbation technique gives [eqn (24)]

$$\frac{K_{\mathrm{TIP}}}{K^{\mathrm{L}}} = 1.0362 - 0.0491\omega. \tag{27}$$

The effective transformation strain technique only calculates the coupling effect of dilatation and moduli mismatch. It does not take into account the individual effect of moduli mismatch. A comparison of eqns (26) and (27) shows that, already for lowest order moduli mismatch, this can make not only a small quantitative, but more importantly a qualitative difference to the predicted shielding effect. Thus, for example, if $\omega = 5$, eqns (26) and (27) give $K_{\text{TIP}}/K^{\text{L}} = 0.742$ and 0.7907, respectively. Comparison with the shielding effect of dilatation alone ($K_{\text{TIP}}/K^{\text{L}} = 1-0.0459\omega = 0.7705$) shows that whereas the present perturbation technique predicts a reduction of about 3%, the effective transformation strain technique predicts an increase of nearly 4%. This needs to be borne in mind when studying the crack-tip shielding in composites, such as ZTA which have large differences in the elastic properties of the transforming and matrix phases.

REFERENCES

Amazigo, J. C. and Budiansky, B. (1988). Steady-state crack growth in supercritically transforming materials. *Int. J. Solids Structures* **24,** 751–755.
Budiansky, B., Hutchinson, J. W. and Lambropoulos, J. C. (1983). Continuum theory of dilatant transformation toughening in ceramics. *Int. J. Solids Structures* **19,** 337–355.
Green, D. J., Hannick, R. J. and Swain, M. V. (1989). *Transformation Toughening of Ceramics.* CRC Press, Boca Raton, FL.
Hill, R. (1965). A self-consistent mechanics of composite materials. *J. Mech. Phys. Solids* **13,** 213–222.
Hutchinson, J. W. (1987). Crack tip shielding by micro-cracking in brittle materials. *Acta Metall.* **25,** 1605–1619.
Karihaloo, B. L. and Huang, X. (1989). Three-dimensional elastic crack tip interactions with pure shear transformation strains. *Int. J. Solids Structures* **25,** 591–607.
Lambropoulos, J. C. (1986). Shear, shape and orientation effects in transformation toughening. *Int. J. Solids Structures* **22,** 1083–1106.
McMeeking, R. M. (1986). Effective transformation strain in binary elastic composites. *J. Am. Ceramics Soc.* **69,** C301–C302.
McMeeking, R. M. and Evans, A. G. (1982). Mechanics of transformation toughening in brittle materials. *J. Am. Ceramics Soc.* **65,** 242–246.
Rice, J. R. (1985). Three dimensional elastic crack tip interactions with transformation strains. *Int. J. Solids Structures* **21,** 781–791.
Rose, L. R. F. (1987). The mechanics of transformation toughening. *Proc. R. Soc. Lond.* **A412,** 169–197.

APPENDIX

The calculation of the integrals in eqn (20) will be given in some detail. For the first term the first step is to simplify the derivative of the displacement:

$$\frac{2}{1-2\nu}\,\text{Re}\left\{\frac{\partial w^0}{\partial z}\right\} = \frac{K^{\text{L}}}{2\mu\sqrt{2\pi}}\left\{\frac{1}{\sqrt{z}}+\frac{1}{\sqrt{\bar{z}}}\right\}+\frac{(1+\nu)c\theta^{\text{T}}}{24\pi(1-\nu)}\int\!\!\int_{\Omega}\{g(z,z_0)+g(\bar{z},z_0)+g(z,\bar{z}_0)+g(\bar{z},\bar{z}_0)\}\,\text{d}A_0,$$

where

$$g(z,z_0) = \frac{1}{\sqrt{zz_0}(\sqrt{z_0}+\sqrt{z})^2}.$$

Integration with respect to x_0 reduces the integral in the above equation to a contour integral around the boundary C:

$$\frac{2}{1-2\nu}\,\text{Re}\left\{\frac{\partial w^0}{\partial z}\right\} = \frac{K^{\text{L}}}{2\mu\sqrt{2\pi}}\left\{\frac{1}{\sqrt{z}}+\frac{1}{\sqrt{\bar{z}}}\right\}-\frac{2(1+\nu)c\theta^{\text{T}}}{24\pi(1-\nu)}\int_C\{G(z,z_0)+G(\bar{z},z_0)+G(z,\bar{z}_0)+G(\bar{z},\bar{z}_0)\}\,\text{d}y_0, \tag{A1}$$

where

$$G(z,z_0) = \frac{1}{\sqrt{z}(\sqrt{z_0}+\sqrt{z})}.$$

This term is to be multiplied by

$$\text{Re}\left\{\frac{\partial h}{\partial z}\right\} = \frac{(1-2\nu)}{8\sqrt{2\pi}(1-\nu)}\left\{\frac{1}{z^{3/2}}+\frac{1}{\bar{z}^{3/2}}\right\}.$$

The first term

$$\frac{4\mu}{(1-2\nu)}\int\!\!\int_{\Omega}\text{Re}\left\{\frac{\partial h}{\partial z}\right\}\text{Re}\left\{\frac{\partial w^0}{\partial z}\right\}\text{d}A \tag{A2}$$

in eqn (20) can now be calculated (after isolating any non-integrable singularities). Some of the integrals can be evaluated exactly analytically for the appropriate transformation zone boundary (15) the rest can be reduced to contour integrals over C which then have to be evaluated numerically. Examples are given below.

The leading term, after isolation of the non-integrable singularity at $z = 0$, is

$$\iint_{\Omega}\left(\frac{1}{z^{3/2}}+\frac{1}{\bar{z}^{3/2}}\right)\left(\frac{1}{\sqrt{z}}+\frac{1}{\sqrt{\bar{z}}}\right)\mathrm{d}A = -3\sqrt{3}+\underline{2\pi}, \tag{A3}$$

where the underlined contribution is from $|z| = \rho$ which is independent of ρ.

There are four terms of the type

$$H(z,z_0) = \iint_{\Omega}\frac{1}{z^{3/2}}\int_C G(z,z_0)\,\mathrm{d}y_0\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_C \mathrm{d}y\int_C \mathrm{d}y_0\left\{\frac{2}{z_0^{3/2}}\ln\left(\frac{\sqrt{z}}{\sqrt{z}+\sqrt{z_0}}\right)+\frac{2}{z_0\sqrt{z}}-\frac{1}{z\sqrt{z_0}}\right\},$$

and when these four terms are combined the non-logarithmic part of the integral can be evaluated exactly (again after isolating the point $z = 0$ with $|z| = \rho$), and the remaining part evaluated numerically. The result is

$$H(z,z_0)+H(\bar{z},z_0)+H(z,\bar{z}_0)+H(\bar{z},\bar{z}_0) = B\left\{A_1+6\sqrt{3}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)-\underline{(5\sqrt{3}\pi+\tfrac{4}{3}\pi^2)}\right\}, \tag{A4}$$

where $B = \{2(1+\nu)K^{\mathrm{L}}\}/\{3\sqrt{2\pi}\sigma_{\mathrm{m}}^{\mathrm{c}}\}$. The numerical factor A_1 $(= -1.1188)$ is the contribution from the logarithmic terms in the integrand which have been evaluated numerically, whereas the underlined term is again the contribution from $|z| = \rho$ which is independent of ρ.

The four terms remaining in this integral have the form

$$H_1(z,z_0) = \iint_{\Omega}\frac{1}{\bar{z}^{3/2}}\int_C G(z,z_0)\,\mathrm{d}y_0\,\mathrm{d}x\,\mathrm{d}y$$
$$= -\int_C \mathrm{d}y\int_C \mathrm{d}y_0\left\{\frac{2(\sqrt{zz_0}-z+\bar{z})}{\sqrt{\bar{z}}(z-\bar{z})(z_0-z+\bar{z})}\right.$$
$$\left.-\frac{2}{(z_0-z+\bar{z})^{3/2}}\ln\left[\frac{2\sqrt{\bar{z}(z_0-z+\bar{z})}-2(z-\bar{z})-2\sqrt{zz_0}}{\sqrt{z}+\sqrt{z_0}}\right]\right\}.$$

When these four terms are combined the apparent singularity on the x-axis when $(z-\bar{z}) = 0$ is removed. The integrals are now integrated numerically to give

$$H_1(z,z_0)+H_1(\bar{z},z_0)+H_1(z,\bar{z}_0)+H_1(\bar{z},\bar{z}_0) = \frac{2(1+\nu)K^{\mathrm{L}}}{3\sqrt{2\pi}\sigma_{\mathrm{m}}^{\mathrm{c}}}A_2, \tag{A5}$$

where the numerical factor $A_2 = -1.6915$.

The second term in eqn (20) will now be evaluated. As above, the derivative $\partial w^0/\partial\bar{z}$ calculated from eqn (19) can be simplified by integration with respect to x_0. The result is

$$\frac{\partial w^0}{\partial\bar{z}} = \frac{K^{\mathrm{L}}}{8\mu\sqrt{2\pi}}\left(\frac{z-\bar{z}}{\bar{z}^{3/2}}\right)-\frac{(1+\nu)c\theta^{\mathrm{T}}}{24\pi(1-\nu)}\int_C\left[\frac{4}{\bar{z}-\bar{z}_0}\right.$$
$$\left.+\frac{z-\bar{z}}{2}\left\{\frac{1}{\sqrt{\bar{z}^3 z_0}}+\frac{1}{\sqrt{\bar{z}^3\bar{z}_0}}-\frac{1}{\sqrt{\bar{z}z_0}(\sqrt{\bar{z}}+\sqrt{z_0})^2}-\frac{1}{\sqrt{\bar{z}\bar{z}_0}(\sqrt{\bar{z}}+\sqrt{\bar{z}_0})^2}\right\}\right]\mathrm{d}y_0. \tag{A6}$$

This term is to be multiplied by

$$\frac{\partial\bar{h}}{\partial z} = -\frac{2}{16\sqrt{2\pi}(1-\nu)}\left(\frac{z-\bar{z}}{z^{5/2}}\right).$$

The second term in eqn (20)

$$2\mu\iint_{\Omega}2\,\mathrm{Re}\left(\frac{\partial\bar{h}}{\partial z}\frac{\partial w^0}{\partial\bar{z}}\right)\mathrm{d}A$$

can now be calculated. Some of the integrals can again be evaluated analytically for the zone boundary (15) while the rest can be reduced to contour integrals over C which can then be evaluated numerically.

After an integration with respect to x, the leading term reduces to

$$G_1 = 2\,\mathrm{Re}\iint_{\Omega} \frac{(z-\bar{z})^2}{\bar{z}^{3/2} z^{5/2}}\,\mathrm{d}A$$
$$= -\mathrm{Re}\int_C \left(\frac{2}{z-\bar{z}}\right)\left\{\sqrt{\frac{z}{\bar{z}}}+2\sqrt{\frac{\bar{z}}{z}}-\frac{1}{3}\left(\frac{\bar{z}}{z}\right)^{3/2}\right\}\mathrm{d}y$$
$$= \frac{16}{3}\left[\left(\frac{\sqrt{3}}{2}\right)^3-\frac{\pi}{6}\right]. \tag{A7}$$

The next term must vanish

$$\iint_{\Omega}\mathrm{d}A\int_C 2\,\mathrm{Re}\left(\frac{4}{\bar{z}-\bar{z}_0}\,\frac{z-\bar{z}}{\bar{z}^{5/2}}\right)\mathrm{d}y_0 \equiv 0, \tag{A8}$$

because a non-zero value would imply a contribution to the S.I.F. from phase transformation in the absence of the crack which is clearly absurd. It is easily verified that this integral does indeed vanish. The next two integrals together, may be written as

$$\int_C \frac{G_1}{2}\left(\frac{1}{\sqrt{z_0}}+\frac{1}{\sqrt{\bar{z}_0}}\right)\mathrm{d}y_0 = \sqrt{\frac{3}{2\pi}}\,\frac{(1+\nu)K^{\mathrm{L}}}{\sigma_{\mathrm{m}}^{\mathrm{c}}}\,\frac{G_1}{2}, \tag{A9}$$

where G_1 is given by eqn (A7).

The penultimate term

$$2\,\mathrm{Re}\iint_{\Omega}\mathrm{d}A\int_C \frac{1}{\sqrt{z_0}}\,\frac{(z-\bar{z})^2}{2}\,\frac{\mathrm{d}y_0}{z^{5/2}\sqrt{\bar{z}}(\sqrt{\bar{z}}+\sqrt{z_0})^2} \tag{A10}$$

is to be evaluated numerically and is equal to

$$\frac{2(1+\nu)K^{\mathrm{L}}}{3\sqrt{2\pi}\sigma_{\mathrm{m}}^{\mathrm{c}}}A_3, \tag{A11}$$

where the numerical factor $A_3 = 8.7426$. The last term

$$2\,\mathrm{Re}\iint_{\Omega}\mathrm{d}A\int_C \frac{1}{\sqrt{\bar{z}_0}}\,\frac{(z-\bar{z})^2}{2}\,\frac{\mathrm{d}y_0}{z^{5/2}\sqrt{\bar{z}}(\sqrt{\bar{z}}+\sqrt{\bar{z}_0})^2} \tag{A12}$$

must also be evaluated numerically, but is equal to the previous integral (A11).

This completes the evaluation of all integrals appearing in eqn (20).